Option Valuation in Heston's Stochastic Volatility Model using finite element methods

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Overview

- Heston's Model

- Existence and Uniqueness

- Short Introduction to Finite Elements

- Results

Heston's Model

- spot S

$$\mathrm{d}S(t) = \mu S(t)\,\mathrm{d}t + \sqrt{v(t)}S(t)\,\mathrm{d}W_1(t)$$

- squared volatility v follows a mean reversion process

$$\mathrm{d}v(t) = \kappa\left[\theta - v(t)\right]\mathrm{d}t + \xi\sqrt{v(t)}\,\mathrm{d}W_2(t)$$

- two correlated Wiener processes W_1 and W_2 with correlation coefficient ρ

Domain

- time: $t \in [0, T]$

- variance: $v \in [\textcolor{red}{0}, \infty)$

- spot: $S \in (0, \infty)$

- logarithm of the spot: $\log \frac{S}{K} \in (-\infty, \infty)$

Heston's Partial Differential Equation

- option price $w(t, v, x)$, $x = \log \frac{S}{K}$

$$
\begin{aligned}
0 \;=\; & w_t + \frac{1}{2}\xi^2 v w_{vv} + \rho\xi v w_{vx} + \frac{1}{2} v w_{xx} \\
& + [\kappa(\theta - v) - \lambda v] w_v + (r_d - r_f - \frac{1}{2} v) w_x \\
& - r_d w
\end{aligned}
$$

$$
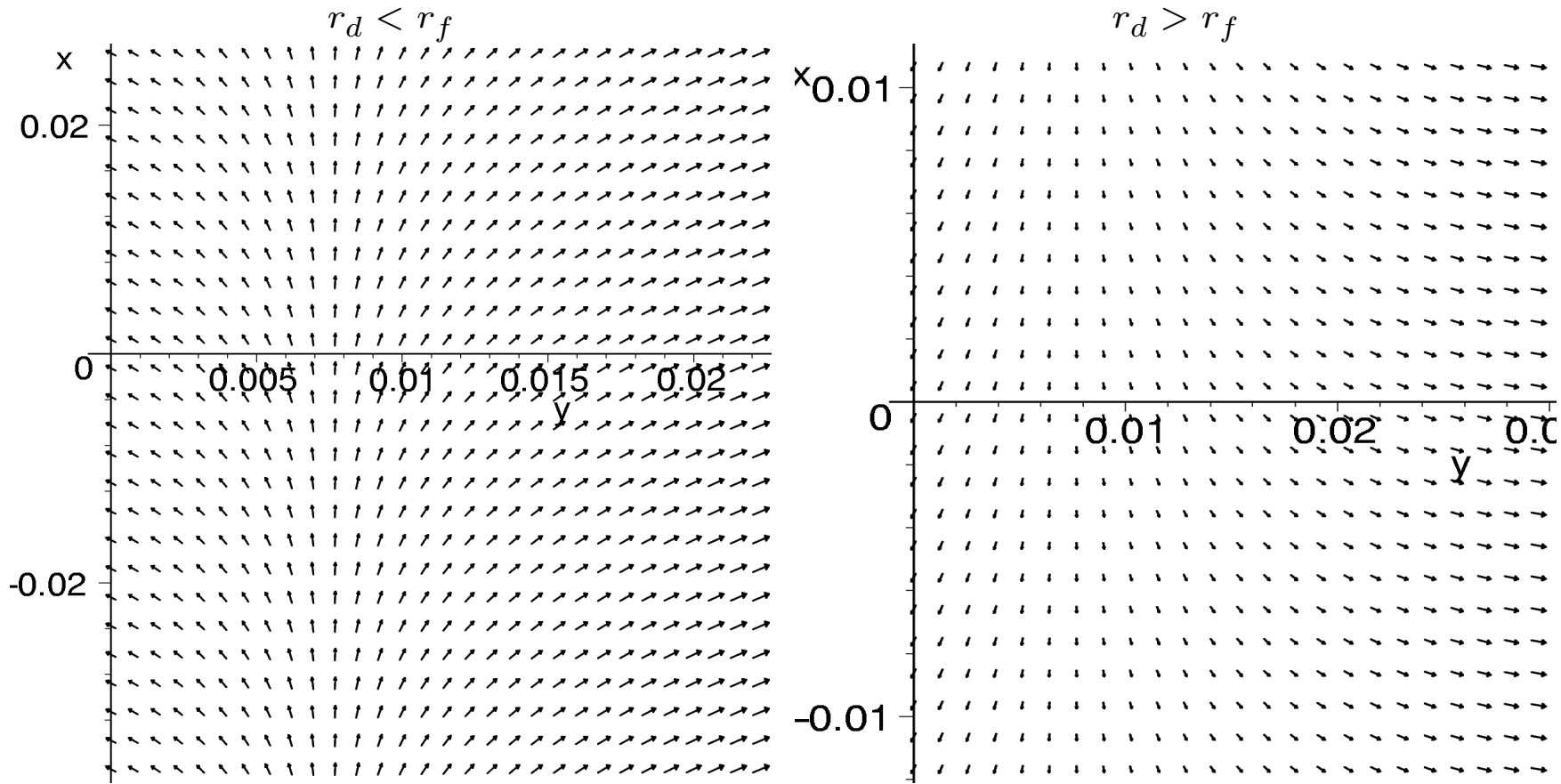w(T, v, x) = g(Ke^x)
$$

- 2nd order parabolic PDE

Heston's PDE

$$w_t + \underbrace{\nabla \cdot A\nabla w}_{\text{diffusion}} - \underbrace{b \cdot \nabla w}_{\text{convection}} - \underbrace{cw}_{\text{reaction}} = 0$$

$$A = \frac{1}{2}v \begin{bmatrix} \xi^2 & \rho\xi \\ \rho\xi & 1 \end{bmatrix} \qquad b = \begin{bmatrix} -\kappa(\theta - v) + \lambda v + \frac{1}{2}\xi^2 \\ -(r_d - r_f) + \frac{1}{2}v + \frac{1}{2}\xi\rho \end{bmatrix}$$

$$c = r_d \qquad \nabla = \begin{bmatrix} \partial_v \\ \partial_x \end{bmatrix}$$

$\rightarrow$variational

Direction of the Flow

$r_d < r_f$

$r_d > r_f$

→meshes

Existence, Uniqueness?

- unbounded domain: $[0, T) \times [0, \infty) \times (-\infty, \infty)$

Existence, Uniqueness?

- unbounded domain: $[0, T) \times [0, \infty) \times (-\infty, \infty)$

- to be investigated:

  – existence and uniqueness of a solution
  – numerical methods to find the solution

Bounded Domain

- choose a bounded domain:

$$Q_t = [0, T] \times \underbrace{(v_{min}, v_{max}) \times (x_{min}, x_{max})}_{\Omega}$$

Bounded Domain

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$$Q_t \;=\; [0,T] \times \underbrace{(v_{min}, v_{max}) \times (x_{min}, x_{max})}_{\Omega}$$

- adequate boundary conditions

  − Dirichlet boundary conditions at 3 sides: $v = v_{min}$, $v = v_{max}$ and $x = x_{min}$
  − Neumann boundary conditions for boundary $x = x_{max}$

Semidiscretization

- discretize the time: $t^k = k\tau$, $k = 0, \ldots, N$

- replace the time derivative by a difference quotient

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- replace the time derivative by a difference quotient

$\Rightarrow$ sequence of elliptic PDEs with initial conditions

Variational Formulation

find functions $w^k(t^k, v, x)$ such that

$$a(w^k, \psi) = \langle F, \psi \rangle \qquad \forall \psi$$

$$a(w^k, \psi) = \int_\Omega A \nabla w^k \cdot \nabla \psi + \frac{1}{2} \int_\Omega (b \cdot \nabla w^k \psi - w^k b \cdot \nabla \psi)$$

$$+ \int_\Omega (c - \frac{1}{2} \nabla \cdot b) w^k \psi + \frac{1}{2} \int_{\Gamma_d} g_1 w^k \psi$$

$$\langle F^k, \psi \rangle = f^k(\psi) + \int_{\Gamma_d} g_2 \psi$$

$\longrightarrow$pde

Existence and Uniqueness

- **Theorem:** The given problem has a unique solution if v_{min} is not too small and if the downward flow is weak.

- **Proof:** Check all assumptions of the lemma of Lax and Milgram.

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Introduction to Finite Elements

- domain $\Omega = (0,1) \times (0,1)$

- boundary $\Gamma = \partial\Omega$, parts Γ_D, Γ_N

- (partial) differential equation

$$
\begin{aligned}
Lu &= f && \text{in } \Omega \\
u &= g_D && \text{on } \Gamma_D \quad \text{Dirichlet b.c.} \\
\frac{\partial u}{\partial \vec{\nu}} &= g_N && \text{on } \Gamma_N \quad \text{Neumann b.c.}
\end{aligned}
$$

Variational Formulation

- multiply with test function and integrate

$$\int_\Omega (Lu)\varphi \,\mathrm{d}x \;=\; \int_\Omega f\varphi \,\mathrm{d}x$$
$$a(u,\varphi) \;=\; \langle F,\varphi\rangle$$

- bilinear form $a(\cdot,\cdot)$

- linear form $\langle F,\cdot\rangle$

Selection of Spaces

- define linear spaces

$$
\begin{aligned}
V &= \{\varphi : a(\varphi, \varphi) < \infty\} \\
V_0 &= \{\varphi \in V : \varphi = 0 \text{ on } \Gamma_D\} \\
V_* &= \{\varphi \in V : \varphi = g_D \text{ on } \Gamma_D\}
\end{aligned}
$$

Task

Find $u \in V_*$ such that

$$a(u, \varphi) \;=\; \langle F, \varphi \rangle \qquad \forall \varphi \in V_0$$

Task

Find $u \in V_*$ such that

$$a(u, \varphi) = \langle F, \varphi \rangle \qquad \forall \varphi \in V_0$$

or with any $\tilde{g} \in V_*$ find $w \in V_0$ such that

$$a(w, \varphi) = \langle F, \varphi \rangle - a(\tilde{g}, \varphi) \qquad \forall \varphi \in V_0$$

note: $w = u - \tilde{g}$.

Discretization

- choose a finite dimensional subspace $V_{0h} \in V_0$

- choose a basis $\{\varphi_i, i = 0, \dots, N\} \subset V_{0h}$

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- choose a finite dimensional subspace $V_{0h} \in V_0$

- choose a basis $\{\varphi_i, i = 0, \ldots, N\} \subset V_{0h}$

- find a discrete solution of the form $w_h = \sum\limits_{j=0}^{N} w_j \varphi_j$

$$a(w_h, \varphi_i) \;=\; \langle F, \varphi_i \rangle - a(\tilde{g}, \varphi_i) \quad i = 0, \ldots, N$$

Discrete task

- solve a linear system of equations

$$
\begin{aligned}
K_h \underline{w_h} &= \underline{F_h} \\
K_{ij} &= a(\varphi_j, \varphi_i) \\
F_i &= \langle F, \varphi_i \rangle - a(\tilde{g}, \varphi_i) \\
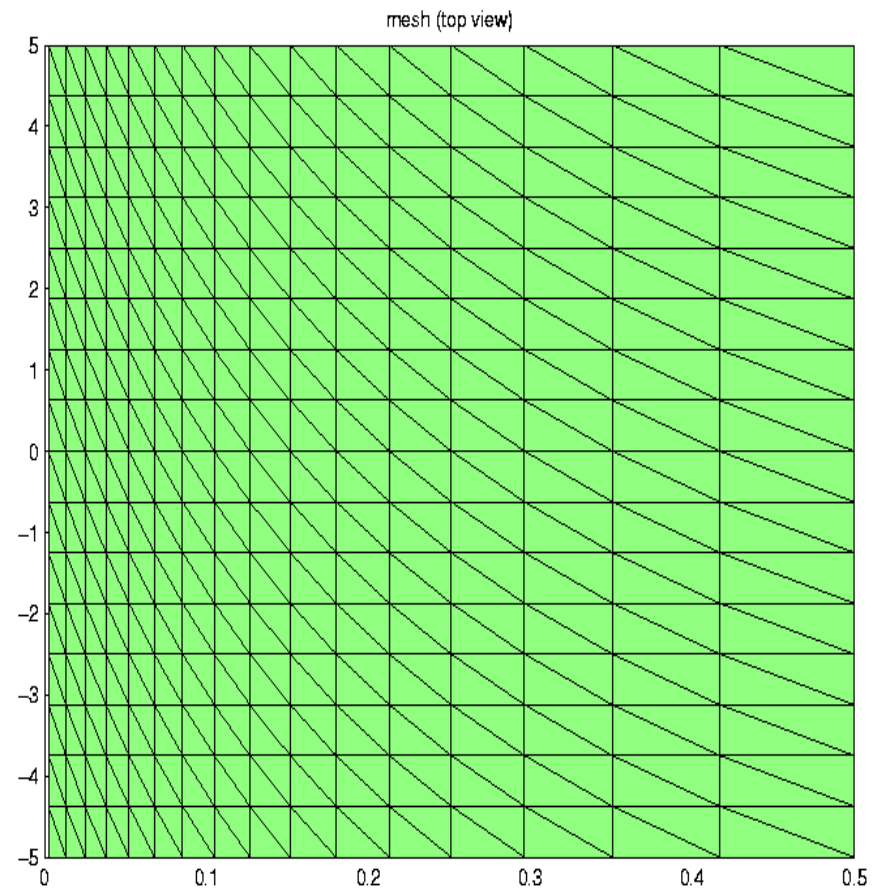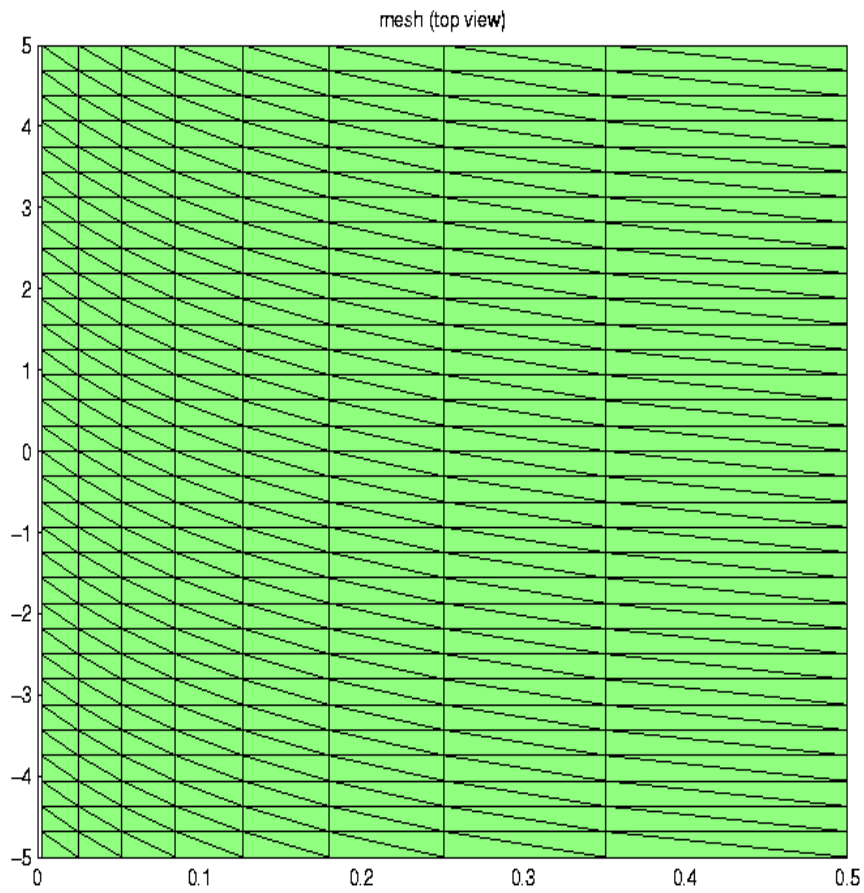i, j &= 0, \ldots, N
\end{aligned}
$$

Finite Element Solution

- starting point: variational formulation

$$a(w, \psi) \;=\; \langle F, \psi \rangle \qquad \forall \psi \in V_0$$

- select a mesh $\mathcal{T}_h$ of the domain Ω

- select test functions φ_i

- compute discrete solution $w_h \in V_{0h} = \mathrm{span}\{\varphi_i\}$
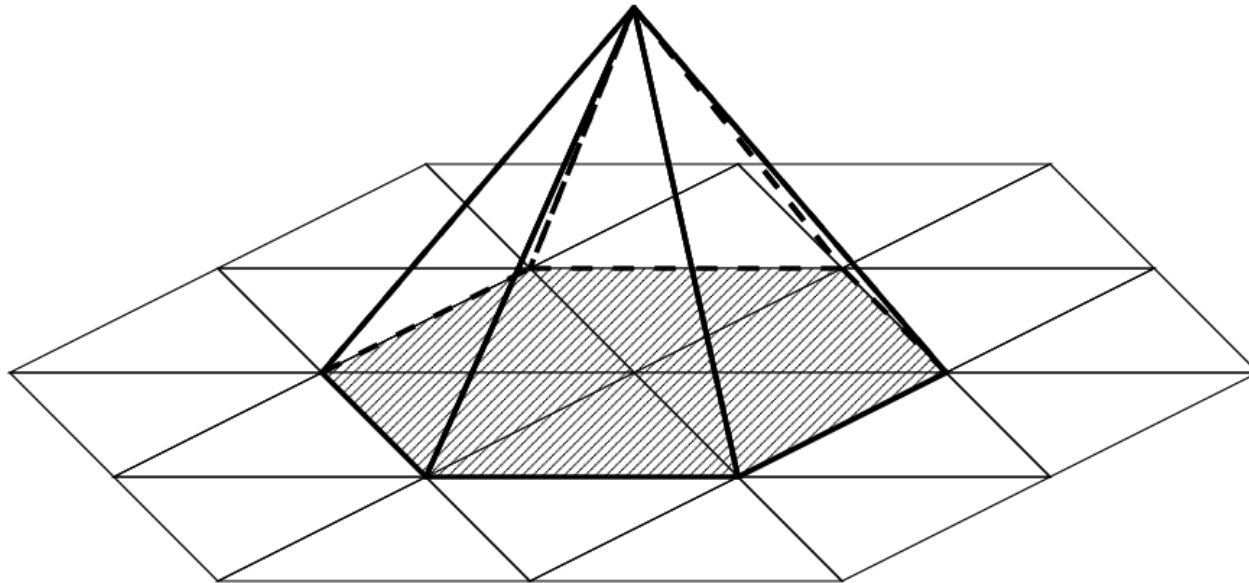
Meshes

→convection

Test Functions
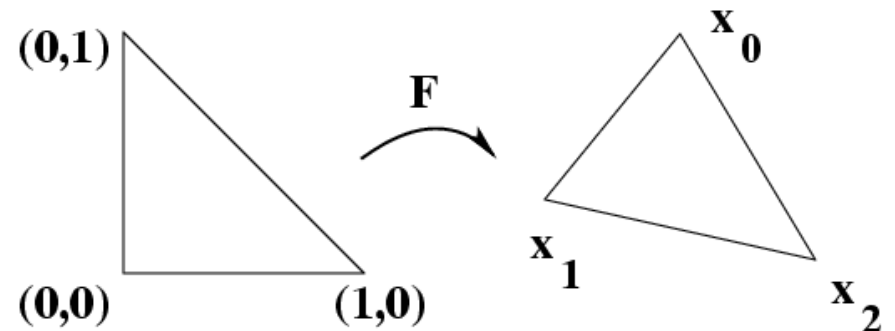
- piecewise linear test function φ_i

Mappings

- map all triangles onto the unit triangle

- do all computations on the unit triangle

- local test functions

$$\hat{\varphi}_0(x,y) = 1 - x - y$$
$$\hat{\varphi}_1(x,y) = x$$
$$\hat{\varphi}_2(x,y) = y$$

Assemble the Matrices

- compute integrals for each triangle

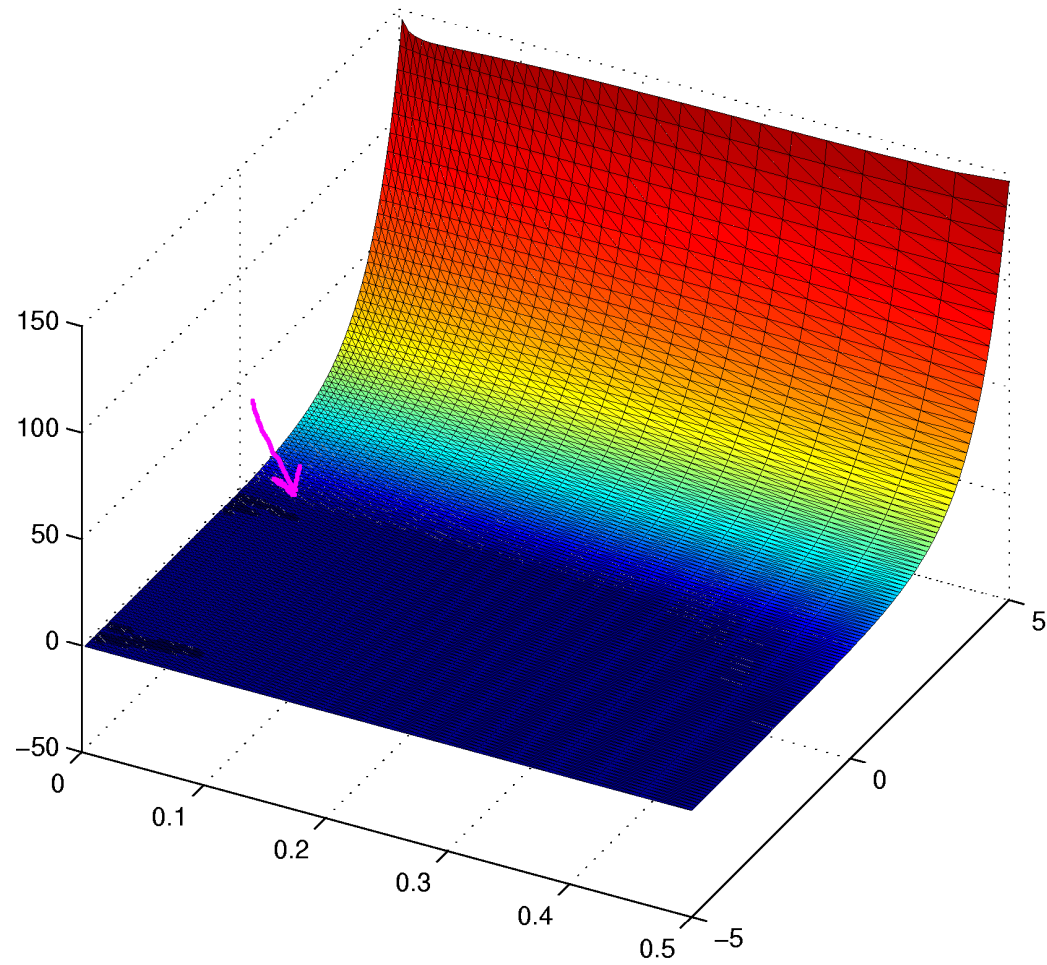$$\Rightarrow \quad K_h \text{ and } F_h$$

- solve $K_h w_h = F_h$

Boundary Conditions

boundary	formula
$v = v_{min}$	$w(t, 0, x) = Ke^{x - r_f(T-t)}\Phi(d_1) - Ke^{-r_d(T-t)}\Phi(d_2)$
$v = v_{max}$	$w(t, v_{max}, x) = Ke^{x - r_f(T-t)}$
$x = x_{min}$	$w = \lambda w(t, v_{max}, x_{min}) + (1 - \lambda)w(t, v_{min}, x_{min})$ $\lambda = \frac{v - v_{min}}{v_{max} - v_{min}}$
$x = x_{max}$	$\frac{\partial}{\partial \nu}w(t, v, x_{max}) := A\nabla w \cdot \vec{n} = \frac{1}{2}vKe^{x - r_f(T-t)}$

Parameters

- $\xi = 0.5$, $\rho = -0.1$, $\lambda = 0$,
  $\kappa = 2.5$, $\theta = 0.06$,
  $r_f = \log(1.048)$, $r_d = \log(1.052)$,
  $K = 1$, $T = 0.25$

Solution

Numbers

method	result	time
Black-Scholes $(\sigma = \sqrt{v_0})$	0.04549	0
Heston, analytic	0.04494	0
FEM, mesh 1	0.04142	65s
FEM, mesh 2	0.04400	327s
FEM, mesh 3	0.04459	2522s

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The End

- Thank you for your attention.

- Where to get these slides:
  `http://www-user.tu-chemnitz.de/~wgu/diplom/diplom.html`